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CONNECTIVITY IN A FUZZY GRAPH

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ABSTRACT
The concept of connectivity play an important role in fuzzy graph theory.In this paper we discuss about cycle connectivity, arc connectivity,node connectivity and complement connectivity cyclic cut vertices,cyclic bridges and cyclically balanced fuzzy graphs. Connectivity of a complement fuzzy graph is analyzed. Also we discussed about connected domination in fuzzy graph using strong arcs.
Key words: cycle connectivity, arc connectivity,node connectivity and complement connectivity cyclic cut vertices,cyclic bridges and cyclically balanced fuzzy graphs.

I.INTRODUCTION TO FUZZYGRAPH

Graph theory is proved to be tremendously useful in modeling the essential features of systems with finite components. Graphical models are used to represent telephone network,railway network, communication problems, traffic network etc. Graph theoretic models can sometimes provide a useful structure upon which analytic techniques can be used. A graph is also used to model a relationship between a given set of objects. Each object is represented by a vertex and the relationship between them is represented by an edge if the relationship is unordered and by means of a directed edge if the objects have an ordered relation between them. Relationship among the objects need not always be precisely defined criteria; when we think of an imprecise concept,the fuzziness arises.

The notion of fuzzy graph was introduced by Rosenfeld in year 1975[2].Fuzzy analogues of manystructures in crispngraph theory,like bridges,cut nodes,connectedness,trees and cycles etc were developed after that. Fuzzy trees were characterized by Sunitha and Vijayakumar [3]. The author have characterized fuzzy trees using its unique maximum spanning tree. A sufficient condition for a node to be a fuzzy cut node is also established.Center problems in fuzzy graph,blocks in fuzzy graphs and properties of self complementary fuzzy graphs were also studied by the same authors. They have obtained a characterization for blocks in fuzzy graphs using the concepts of strongest paths [8].The authors have need the concepts of strong arcs and strong paths.As far as the applications are concerned(information networks,electric circuits, etc.),the reduction of flow between pairs of nodes is more relevant and many frequently occur than the total disruption of the theorem or the disconnection of the entire networks.In this paper we put forward the conditions under which a fuzzy graph and its complement will be connected.

In1965,L.A.Zadeh introduced a mathematical frame work to explain the concept to fun certainty in real life through the publication of a seminar paper.A fuzzy set is defined mathematically by assigning to each possible individual in the universe of discourse a value, representing its grade of membership, which corresponds to the degree, to which that individual is similar or compatible with the concept represented by the fuzzy set.The fuzzy graph introduced by A. Rosen feld using fuzzy relation, represents the relationship between the objects by precisely indicating the level of the relationship between the objects of the given set. Also he coined many fuzzy analogous graph theoretic concepts like bridge, cut vertex and tree. Fuzzy graphs have many more applications in modeling real time systems where the level of information inherent in the system varies with different levels of precision.

The notion of fuzzy set stems from the observation made by[18] L.A.Zadeh (1965) that more often than not, the classes of objects encountered in real physical world do not have precisely defined criterial of membership".This observation emphasis the gap existing between mental representation of reality and usual mathematical representation therefore,which are based onbinary logics,precies numbers,differential

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equations and the like.Classes of objects referred to in [18]K.R. Bhutani introduced the concept of press mordeson introduced fuzzy lines graph.

ORGANIZATION OF THE DESTINATION:

Deals with basic definition in fuzzy graph
Deals with cycle connectivity in fuzzy graph
Deals with complement connectivity in fuzzy graph
Deals with node connectivity and arc connectivity in fuzzy graph
Deals with strong connected dominating in fuzzy graph
Deals with cyclic bridges in fuzzy graph
Deals with cyclically balanced in fuzzy graph

3.1 Cycle Connectivity in Fuzzy Graphs

Theorem 3.1.1: A fuzzy graph G is a fuzzy tree if and only if CC(G) = 0.
Proof: If G is an f-tree, then $CG(u,v)=0CG(u,v)=0$ for every pair of nodes u and v in G. Hence, it follows that $CC(G)=0CC(G)=0$. Conversely, suppose that $CC(G)=0CC(G)=0$. Therefore, $CG(u,v)=0CG(u,v)=0$ for every pair of nodes in G, meaning G has no strong cycles. Consequently, G has no fuzzy cycles, indicating that G is an f-tree.

Proposition 3.1.2: The cycle connectivity of a fuzzy cycle G is the strength of G.
Proof: This follows from the fact that any fuzzy cycle is a strong cycle.
Theorem 3.1.3: Let G be a complete fuzzy graph with nodes $v1,v2,...,vnv1,v2,...,vn$ such that $\sigma(vi)=ti\sigma(vi)=ti$ and $t1\leq t2\leq ...\leq tn-2\leq tn-1\leq tnt1\leq t2\leq ...\leq tn-2\leq tn-1\leq tn$. Then $CC(G)=tn-2CC(G)=tn-2$.

Proof: Assume the conditions of the theorem. Since any three nodes of G are adjacent, any three nodes form a 3-cycle. Additionally, all arcs in a complete fuzzy graph are strong. To find the maximum strength of cycles in G, it is sufficient to find the maximum strength of all 3-cycles in G. Consider a 4-cycle $C=abcdC=abcd$ in G (the case for an n-cycle is similar). Since G is complete, there exist parts of two 3-cycles in C, namely $C1=abcC1=abc$ and $C2=acdC2=acd$. Let the strength $s(C)=ts(C)=t$. For all edges (x, y) in C, $\mu(x,y)\geq t\mu(x,y)\geq t$. In particular, $\mu(a,b)\geq t\mu(a,b)\geq t$ and $\mu(b,c)\geq t\mu(b,c)\geq t$. Since G is a complete fuzzy graph, G has no δ arcs. Thus, $\mu(a,c)\geq \mathrm{Min}\{\mu(a,b),\mu(b,c)\}\geq t\mu(a,c)\geq \mathrm{Min}\{\mu(a,b),\mu(b,c)\}\geq t$. That is, $\mu(a,c)\geq t\mu(a,c)\geq t$. Suppose $\mu(a,c)=t\mu(a,c)=t$, then $s(C1)=s(C2)=s(C)=ts(C1)=s(C2)=s(C)=t$. Suppose $\mu(a,c)>t\mu(a,c)>t$, then since $s(C)=ts(C)=t$, at least one of $C1C1$ or $C2C2$ will have strength equal to t. In either case, $s(C)=\mathrm{Min}\{s(C1),s(C2)\}s(C)=\mathrm{Min}\{s(C1),s(C2)\}$. Thus, the strength of a 4-cycle is nothing but the strength of a 3-cycle in G. Among all 3-cycles, the 3-cycle formed by three nodes with maximum node strength will have the maximum strength. Thus, the cycle $C=vn-2vn-1vnvn-2C=vn-2vn-1vnvn-2$ is a cycle with maximum strength in G. Also, the strength of $C=tn-2\wedge tn-1\wedge tn=tn-2C=tn-2\wedge tn-1\wedge tn=tn-2$, where $\wedge\wedge$ stands for the minimum. Thus, $CC(G)=tn-2CC(G)=tn-2$.

III. Connectivity in a Fuzzy Graph

Definition 2.14: A fuzzy graph GG is connected if $\mu\infty(u,v)>0\mu\infty(u,v)>0$ for all $u,v\in\sigma*u,v\in\sigma*$. An arc $(x,y)(x,y)$ is a strong arc if $\mu(x,y)\geq\mu\infty(x,y)\mu(x,y)\geq\mu\infty(x,y)$. A node is an isolated node if $\mu(x,y)=0\mu(x,y)=0$ for all $y\neq xy\square=x$.
Definition 2.15: $G=(\sigma,\mu)G=(\sigma,\mu)$ is a fuzzy cycle if $(\sigma*,\mu*)(\sigma*,\mu*)$ is a cycle and there does not exist a unique $(x,y)\in\mu*(x,y)\in\mu*$ such that $\mu(x,y)=\min\{\mu(u,v)/(u,v)\in\mu*\}\mu(x,y)=\min\{\mu(u,v)/(u,v)\in\mu*\}$.
Definition 2.16: Let $G=(\sigma,\mu)G=(\sigma,\mu)$ be a fuzzy graph. The complement of GG is defined as $Gc=(\sigma c,\mu c)Gc=(\sigma c,\mu c)$, where $\mu c(x,y)=\sigma(x)\wedge\sigma(y)-\mu(x,y)\mu c(x,y)=\sigma(x)\wedge\sigma(y)-\mu(x,y)$ for all $x,y\in Sx,y\in S$.
Definition 2.17: The $\mu\mu$-complement of GG is denoted as $G\mu=(\sigma\mu,\mu\mu)G\mu=(\sigma\mu,\mu\mu)$, where $\sigma\mu=\sigma\sigma\mu=\sigma$ and $\mu\mu(u,v)=0\mu\mu(u,v)=0$ if $\mu(u,v)=0\mu(u,v)=0$ and $\mu\mu(u,v)=\sigma(u)\wedge\sigma(v)-\mu(u,v)\mu\mu(u,v)=\sigma(u)\wedge\sigma(v)-\mu(u,v)$ if $\mu(u,v)>0\mu(u,v)>0$.

Definition 2.18: The busy value of a node vv in GG is $D(v)=\sum\sigma(v)\wedge\sigma(vi)D(v)=\sum\sigma(v)\wedge\sigma(vi)$, where $vivi$ are the neighbors of vv. The busy value of GG is $D(G)=\sum D(vi)D(G)=\sum D(vi)$, where $vivi$ are the nodes of GG.
Definition 2.19: A node in GG is a busy node if $\sigma(v)\leq d(v)\sigma(v)\leq d(v)$; otherwise, it is called a free node.
Definition 2.20: A node vv of a fuzzy graph GG is said to be:

- A partial free node if it is a free node in both GG and $G\mu G\mu$.
- A fully free node if it is free in GG but busy in $G\mu G\mu$.
- A partial busy node if it is a busy node in both GG and $G\mu G\mu$.
- A fully busy node if it is busy in GG but free in $G\mu G\mu$.

Definition 2.21: Two nodes of a fuzzy graph are said to be fuzzy independent if there is no strong arc between them.
Definition 2.22: A subset $S'S'$ of SS is said to be fuzzy independent if any two nodes of $S'S'$ are fuzzy independent.
Definition 2.23: A fuzzy graph GG is said to be fuzzy bipartite if the node set SS can be partitioned into two subsets $S1S1$ and $S2S2$ such that $S1S1$ and $S2S2$ are fuzzy independent sets. These sets are called the fuzzy bipartition of SS.
Definition 2.24: A fuzzy matrix is a matrix whose elements take values from the interval [0,1][0,1].
Definition 2.25: A fuzzy graph that has no cycles is called acyclic or a forest. A connected forest is called a fuzzy tree. It is also denoted as an f-tree.
Definition 2.27: Let XX be a cyclic vertex cut of GG. The strong weight of XX is defined as $Sc(X)=\sum\mu(x,y)Sc(X)=\sum\mu(x,y)$, where $\mu(x,y)\mu(x,y)$ is the minimum weight of the strong edges incident on XX.
Definition 2.28: The cyclic vertex connectivity of a fuzzy graph GG, denoted by $kc(G)kc(G)$, is the minimum of the cyclic strong weights of cyclic vertex cuts in GG.
Definition 2.29: A cyclic edge cut of a fuzzy graph $G=(\sigma,\mu)G=(\sigma,\mu)$ is a set of edges $Y\subseteq\mu*Y\subseteq\mu*$ such that $CC(G-Y)<CC(G)CC(G-Y)<CC(G)$, provided $CC(G)>0CC(G)>0$. $k'(G)k'(G)$ is the minimum of the strong edge cuts in GG.
Definition 2.30: Let $G=(\sigma,\mu)G=(\sigma,\mu)$ be a fuzzy graph. The strong weight of a cyclic edge cut YY of GG is defined as $S'(Y)=\sum\mu(ei)S'(Y)=\sum\mu(ei)$, where $eiei$ is a strong edge of YY.
Definition 2.31: The cyclic edge connectivity of a fuzzy graph GG is denoted by the weights of cyclic edge cuts in GG.
Definition 2.32: A fuzzy graph is cyclically balanced if it has no cyclic fuzzy cut vertices and no cyclic fuzzy cut bridges.

III. Connectivity in a Fuzzy Graph

Cycle Connectivity in Fuzzy Graphs

Theorem 3.1.1: A fuzzy graph GG is a fuzzy tree if and only if $CC(G)=0CC(G)=0$.
Proof: If GG is an f-tree, then $CG(u,v)=0CG(u,v)=0$ for every pair of nodes uu and vv in GG. Hence, it follows that $CC(G)=0CC(G)=0$. Conversely, suppose that $CC(G)=0CC(G)=0$. Hence, $CG(u,v)=0CG(u,v)=0$ for every pair of nodes in GG. That means GG has no strong cycles. Therefore, GG has no fuzzy cycles, and thus it follows that GG is an f-tree.
Proposition 3.1.2: The cycle connectivity of a fuzzy cycle GG is the strength of GG.
Proof: This follows from the fact that any fuzzy cycle is a strong cycle.
Theorem 3.1.3: Let GG be a complete fuzzy graph with nodes $v1,v2,...,vnv1,v2,...,vn$ such that $\sigma(vi)=ti\sigma(vi)=ti$ and $t1\leq t2\leq...\leq tn-2\leq tn-1\leq tnt1\leq t2\leq...\leq tn-2\leq tn-1\leq tn$. Then $CC(G)=tn-2CC(G)=tn-2$.
Proof: Assume the conditions of the theorem. Since any three nodes of GG are adjacent, any three nodes are in a 3-cycle. Also, all arcs in a complete fuzzy graph are strong. Thus, to find the maximum strength of cycles in GG, it is sufficient to find the maximum strength of all 3-cycles in GG. Consider a 4-cycle $C=abcdaC=abcda$ in GG (the case of an n-cycle is similar). Since GG is complete, there exist parts of two 3-cycles in CC, namely $C1=abcaC1=abca$ and $C2=acdaC2=acda$. Let the strength $s(C)=ts(C)=t$. For all edges $(x,y)(x,y)$ in CC, $\mu(x,y)\geq t\mu(x,y)\geq t$. In particular, $\mu(a,b)\geq t\mu(a,b)\geq t$ and $\mu(b,c)\geq t\mu(b,c)\geq t$. Since GG is a complete fuzzy graph, GG has no δ arcs. Thus, $\mu(a,c)\geq \mathrm{Min}\{\mu(a,b),\mu(b,c)\}\geq t\mu(a,c)\geq \mathrm{Min}\{\mu(a,b),\mu(b,c)\}\geq t$. That is, $\mu(a,c)\geq t\mu(a,c)\geq t$. Suppose $\mu(a,c)=t\mu(a,c)=t$, then $s(C1)=s(C2)=s(C)=ts(C1)=s(C2)=s(C)=t$. Suppose $\mu(a,c)>t\mu(a,c)>t$, then since $s(C)=ts(C)=t$, at least one of $C1C1$ or $C2C2$ will have strength equal to tt. In

either case, $s(C)$=Min$\{s(C1),s(C2)\}s(C)$=Min$\{s(C1),s(C2)\}$. Thus, the strength of a 4-cycle is nothing but the strength of a 3-cycle in GG. Among all 3-cycles, the 3-cycle formed by three nodes with maximum node strength will have the maximum strength. Thus, the cycle $C=vn-2vn-1vnvn-2C=vn-2vn-1vnvn-2$ is a cycle with maximum strength in GG. Also, the strength of CC is $tn-2\wedge tn-1\wedge tn=tn-2tn-2\wedge tn-1\wedge tn=tn-2$, where $\wedge\wedge$ stands for the minimum. Thus, $CC(G)=tn-2CC(G)=tn-2$.

Proposition 3.1.4: In a fuzzy graph, if the arc $(u,v)(u,v)$ is a cyclic bridge, then both uu and vv are cyclic cut nodes.

Proof: Let $G(\sigma,\mu)G(\sigma,\mu)$ be a fuzzy graph and $(u,v)(u,v)$ be a cyclic bridge in GG. Then $CC(G-(u,v))<CC(G)CC(G-(u,v))<CC(G)$. Hence $CC(G-u)\leq CC(G-(u,v))<CC(G)CC(G-u)\leq CC(G-(u,v))<CC(G)$ and $CC(G-v)\leq CC(G-(u,v))<CC(G)CC(G-v)\leq CC(G-(u,v))<CC(G)$. Thus, uu and vv are cyclic cut nodes.

Proposition 3.1.5: Let GG be a fuzzy graph such that $G*G*$ is a cycle. Then,

- GG has no cyclic cut nodes or cyclic bridges if GG is a fuzzy tree.
- All arcs in GG are cyclic bridges and all nodes in GG are cyclic cut nodes if GG is a strong cycle.

Proof: This follows from the fact that a fuzzy tree has no strong cycles. If GG is a strong cycle, then $CC(G)=CC(G)=$ the strength of GG. The removal of any arc or node will reduce its cycle connectivity to 0.

Theorem 3.1.6: Let $G=(\sigma,\mu)G=(\sigma,\mu)$ be a complete fuzzy graph with $|\sigma*|\geq 4|\sigma*|\geq 4$. Let $v1,v2,...,vn\in\sigma*v1,v2,...,vn\in\sigma*$ and $\sigma(vi)=ci\sigma(vi)=ci$ for $i=1,2,...,ni=1,2,...,n$ and $c1\leq c2\leq...\leq cnc1\leq c2\leq...\leq cn$. Then GG has a cyclic cut node (or cyclic bridge) if and only if $cn-3<cn-2cn-3<cn-2$. Further, there exist three cyclic cut nodes (or cyclic bridges) in a complete fuzzy graph (if they exist).

Proof: Let $v1,v2,...,vn\in\sigma*v1,v2,...,vn\in\sigma*$ and $\sigma(vi)=ci\sigma(vi)=ci$ for $i=1,2,...,ni=1,2,...,n$ and $c1\leq c2\leq...\leq cnc1\leq c2\leq...\leq cn$. Suppose that GG has a cyclic cut node uu. Then $CC(G-u)<CC(G)CC(G-u)<CC(G)$. That is, uu belongs to a unique cycle CC with $\alpha=\alpha=$ the strength of CC greater than the strength of any other cycle $C'C'$ in GG. Since $c1\leq c2\leq...\leq cnc1\leq c2\leq...\leq cn$, it follows that the strength of the cycle $vn-2vn-1vnvn-2vn-1vn$ is $\alpha\alpha$. Hence $u\in\{vn-2,vn-1,vn\}u\in\{vn-2,vn-1,vn\}$...(1). To prove $cn-3<cn-2cn-3<cn-2$, suppose not. That is, $cn-3=cn-2cn-3=cn-2$. Then $C1=vnvn-1vn-2C1=vnvn-1vn-2$ and $C2=vnvn-1vn-3C2=vnvn-1vn-3$ have the same strength, and hence the removal of $vn-2,vn-1,vn-2,vn-1$, or $vnvn$ will not reduce $CC(G)CC(G)$, which is a contradiction to (1). Hence, $cn-3<cn-2cn-3<cn-2$. Conversely, suppose that $cn-3<cn-2cn-3<cn-2$. To prove GG has a cyclic cut node. Since $cn\geq cn-1\geq cn-2cn\geq cn-1\geq cn-2$ and $cn-2>cn-3cn-2>cn-3$, all cycles of GG have strength less than that of the strength of $vnvn-1vn-2vnvn-1vn-2$. Hence, the deletion of $vn,vn-1,vn,vn-1$, or $vn-2vn-2$ will reduce the cycle connectivity of GG. Hence, $vn,vn-1,vn,vn-1$, and $vn-2vn-2$ are cyclic cut nodes of GG.

Theorem 3.1.7: For a complete fuzzy graph GG, $kc(G)\leq k(G)kc(G)\leq k(G)$.

Proof: Given a complete fuzzy graph GG with vertices $v1,v2,...,vnv1,v2,...,vn$ such that $ds(v1)\leq ds(v2)\leq...\leq ds(vn)ds(v1)\leq ds(v2)\leq...\leq ds(vn)$. Let $v1v1$ be a vertex such that $ds(v1)=\delta s(G)ds(v1)=\delta s(G)$.

Case I: If $v1v1$ is a cyclic cut vertex. Here, $V=\{v1\}V=\{v1\}$ is a cyclic cut set of GG. Therefore, $Sc(V)=\min\{\mu(v1,vi)\}Sc(V)=\min\{\mu(v1,vi)\}$ for $i=\{2,...,n\}\leq\sum\mu(v1,vi)=\delta s(G)i=\{2,...,n\}\leq\sum\mu(v1,vi)=\delta s(G)$. Now, since $kc(G)=\min\{Sc(V)\},kc(G)=\min\{Sc(V)\}$, where VV is a cyclic cut of GG, we have $kc(G)\leq Sc(V)\leq\delta s(G)=k(G)kc(G)\leq Sc(V)\leq\delta s(G)=k(G)$.

Case II: If $v1v1$ is not a cyclic cut vertex. Let $F=\{u1,u2,...,ut\}F=\{u1,u2,...,ut\}$ be a cyclic cut set such that $Sc(F)=kc(G)Sc(F)=kc(G)$. Now, $kc(G)=Sc(F)kc(G)=Sc(F)$ $=\sum\min\{\mu(ui,uj)\},\forall ui,uj\in\sigma*$ for $j\neq i,j=1,2,...,n=\sum\min\{\mu(ui,uj)\},\forall ui,uj\in\sigma*$ for j□$=i,j=1,2,...,n$ $\leq ds(v1)\leq ds(v1)$ $=\delta s(G)=\delta s(G)$ $=k(G)=k(G)$

Corollary 3.1.8: A vertex in a fuzzy graph is a cyclic cut vertex if and only if it is a common vertex of all strong cycles with maximum strength.

Proof: Let GG be a fuzzy graph. Let ww be a cyclic cut vertex of GG. Then $CC(G-w)<CC(G)CC(G-w)<CC(G)$, i.e., max$\{S(C)$, where C is a strong cycle in $G-w\}<$max$\{s(C')$, where C' is a strong cycle in $G\}$max$\{S(C)$, where C is a strong cycle in $G-w\}<$max$\{s(C')$, where C' is a strong cycle in $G\}$. Therefore, all strong cycles in GG with maximum strength are removed by the deletion of ww. Hence, ww is a common vertex of all strong cycles with maximum strength. This results in the reduction of the cycle connectivity of GG. Thus, ww is a cyclic cut vertex of GG.

Theorem 3.1.9

Theorem 3.1.9: Let $G=(\sigma,\mu)G=(\sigma,\mu)$ be a fuzzy graph. Then no cyclic cut vertex is a fuzzy end vertex of GG.

Proof: Let $G=(\sigma,\mu)G=(\sigma,\mu)$ be a fuzzy graph. Let ww be a cyclic cut vertex of GG. Then ww lies on a strong cycle with maximum strength in GG. Clearly, ww has at least two strong neighbors in GG. Hence, ww cannot be a fuzzy end vertex of GG.

Conversely, if ww is a fuzzy end vertex of GG with $|Ns(w)|=1|Ns(w)|=1$, where $Ns(w)Ns(w)$ is the neighboring set of ww, then ww cannot lie on a strong cycle in GG. This implies that ww is not a cyclic cut vertex of the fuzzy graph GG.

Theorem 3.1.10

Theorem 3.1.10: Let $G=(\sigma,\mu)G=(\sigma,\mu)$ be a complete fuzzy graph with $|\sigma*|\geq 4|\sigma*|\geq 4$. Suppose $v1,v2,...,vn\in\sigma*v1,v2,...,vn\in\sigma*$ and $\sigma(vi)=ci\sigma(vi)=ci$ for $i=1,2,...,ni=1,2,...,n$ and $c1\leq c2\leq...\leq cnc1\leq c2\leq...\leq cn$. Then GG is cyclically balanced if and only if $cn-3=cn-2cn-3=cn-2$.

Proof: Let $v1,v2,...,vn\in\sigma*v1,v2,...,vn\in\sigma*$ and $\sigma(vi)=ci\sigma(vi)=ci$ for $i=1,2,...,ni=1,2,...,n$ and $c1\leq c2\leq...\leq cnc1$ $\leq c2\leq...\leq cn$. Suppose GG is cyclically balanced. To prove that $cn-3=cn-2cn-3=cn-2$, assume the contrary, that $cn-3<cn-2cn-3<cn-2$. Since $cn-2\leq cn-1\leq cncn-2\leq cn-1\leq cn$ and $cn-3<cn-2cn-3<cn-2$, all cycles of GG have strength less than the strength of the cycle $vnvn-1vn-2vnvnvn-1vn-2vn$. Hence, the deletion of any of the three vertices $vnvn$, $vn-1vn-1$, or $vn-2vn-2$ reduces the cycle connectivity of GG. Therefore, $vnvn$, $vn-1vn-1$, and $vn-2vn-2$ are cyclic cut vertices of GG, which contradicts the fact that GG is cyclically balanced.

Conversely, suppose that $cn-3=cn-2cn-3=cn-2$. Then the cycles $C1=vnvn-1vn-2vnC1=vnvn-1vn-2vn$ and $C2=vnvn-1vn-3vnC2=vnvn-1vn-3vn$ have the same strength, and hence the removal of $vnvn$, $vn-1vn-1$, or $vn-2vn-2$ will not reduce the cyclic connectivity of GG. That is, there does not exist any cyclic fuzzy cut vertex in GG. Hence, the fuzzy graph $G=(\sigma,\mu)G=(\sigma,\mu)$ is cyclically balanced.

III. Connectivity in a Fuzzy Graph

Cycle Connectivity in Fuzzy Graphs

Theorem 3.1.1: A fuzzy graph GG is a fuzzy tree if and only if $CC(G)=0CC(G)=0$.

Proof: If GG is an f-tree, then $CG(u,v)=0CG(u,v)=0$ for every pair of nodes uu and vv in GG. Hence, it follows that $CC(G)=0CC(G)=0$. Conversely, suppose that $CC(G)=0CC(G)=0$. Hence, $CG(u,v)=0CG(u,v)=0$ for every pair of nodes in GG. That means GG has no strong cycles. Therefore, GG has no fuzzy cycles, and thus it follows that GG is an f-tree.

Proposition 3.1.2: The cycle connectivity of a fuzzy cycle GG is the strength of GG.

Proof: This follows from the fact that any fuzzy cycle is a strong cycle.

Theorem 3.1.3: Let GG be a complete fuzzy graph with nodes $v1,v2,...,vnv1,v2,...,vn$ such that $\sigma(vi)=ti\sigma(vi$ $)=ti$ and $t1\leq t2\leq...\leq tn-2\leq tn-1\leq tnt1\leq t2\leq...\leq tn-2\leq tn-1\leq tn$. Then $CC(G)=tn-2CC(G)=tn-2$.

Proof: Assume the conditions of the theorem. Since any three nodes of GG are adjacent, any three nodes are in a 3-cycle. Also, all arcs in a complete fuzzy graph are strong. Thus, to find the maximum strength of cycles in GG, it is sufficient to find the maximum strength of all 3-cycles in GG. Consider a 4-cycle $C=abcdC=abcd$ in GG (the case of an n-cycle is similar). Since GG is complete, there exist parts of two 3-cycles in CC, namely $C1=abcaC1=abca$ and $C2=acdaC2=acda$. Let the strength $s(C)=ts(C)=t$. For all edges $(x,y)(x,y)$ in CC, $\mu(x,y)\geq t\mu(x,y)\geq t$. In particular, $\mu(a,b)\geq t\mu(a,b)\geq t$ and $\mu(b,c)\geq t\mu(b,c)\geq t$. Since GG is a complete fuzzy graph, GG has no δ arcs. Thus, $\mu(a,c)\geq \mathrm{Min}\{\mu(a,b),\mu(b,c)\}\geq t\mu(a,c)\geq \mathrm{Min}\{\mu(a,b),\mu(b,c)\}\geq t$. That is, $\mu(a,c)\geq t\mu(a,c)\geq t$. Suppose $\mu(a,c)=t\mu(a,c)=t$, then $s(C1)=s(C2)=s(C)=ts(C1)=s(C2)=s(C)=t$. Suppose $\mu(a,c)>t\mu(a,c)>t$, then since $s(C)=ts(C)=t$, at least one of $C1C1$ or $C2C2$ will have strength equal to tt. In either case, s(C) = \text{Min} \{ s(C_1), s(C_2) \}. Thus, the strength of a 4-cycle is nothing but the strength of a 3-cycle in \(G. Among all 3-cycles, the 3-cycle formed by three nodes with maximum node strength will have the maximum strength. Thus, the cycle $C=vn-2vn-1vnvn-2C=vn-2vn-1vnvn-2$ is a cycle with maximum strength in GG. Also, the strength of CC is $tn-2\wedge tn-1\wedge tn=tn-2tn-2\wedge tn-1\wedge tn=tn-2$, where $\wedge\wedge$ stands for the minimum. Thus, $CC(G)=tn-2CC(G)=tn-2$.

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Proposition 3.1.4: In a fuzzy graph, if the arc $(u,v)(u,v)$ is a cyclic bridge, then both uu and vv are cyclic cut nodes.

Proof: Let $G(\sigma,\mu)G(\sigma,\mu)$ be a fuzzy graph and $(u,v)(u,v)$ be a cyclic bridge in GG. Then $CC(G-(u,v))<CC(G)CC(G-(u,v))<CC(G)$. Hence $CC(G-u)\leq CC(G-(u,v))<CC(G)CC(G-u)\leq CC(G-(u,v))<CC(G)$ and $CC(G-v)\leq CC(G-(u,v))<CC(G)CC(G-v)\leq CC(G-(u,v))<CC(G)$. Thus, uu and vv are cyclic cut nodes.

Proposition 3.1.5: Let GG be a fuzzy graph such that $G*G*$ is a cycle. Then,

- GG has no cyclic cut nodes or cyclic bridges if GG is a fuzzy tree.
- All arcs in GG are cyclic bridges and all nodes in GG are cyclic cut nodes if GG is a strong cycle.

Proof: This follows from the fact that a fuzzy tree has no strong cycles. If GG is a strong cycle, then $CC(G)=CC(G)=$ the strength of GG. The removal of any arc or node will reduce its cycle connectivity to 0.

Theorem 3.1.6: Let $G=(\sigma,\mu)G=(\sigma,\mu)$ be a complete fuzzy graph with $|\sigma*|\geq 4|\sigma*|\geq 4$. Let $v1,v2,...,vn\in\sigma*v1,v2,...,vn\in\sigma*$ and $\sigma(vi)=ci\sigma(vi)=ci$ for $i=1,2,...,ni=1,2,...,n$ and $c1\leq c2\leq...\leq cnc1\leq c2\leq...\leq cn$. Then GG has a cyclic cut node (or cyclic bridge) if and only if $cn-3<cn-2cn-3<cn-2$. Further, there exist three cyclic cut nodes (or cyclic bridges) in a complete fuzzy graph (if they exist).

Proof: Let $v1,v2,...,vn\in\sigma*v1,v2,...,vn\in\sigma*$ and $\sigma(vi)=ci\sigma(vi)=ci$ for $i=1,2,...,ni=1,2,...,n$ and $c1\leq c2\leq...\leq cnc1\leq c2\leq...\leq cn$. Suppose that GG has a cyclic cut node uu. Then $CC(G-u)<CC(G)CC(G-u)<CC(G)$. That is, uu belongs to a unique cycle CC with $\alpha=\alpha=$ the strength of CC greater than the strength of any other cycle $C'C'$ in GG. Since $c1\leq c2\leq...\leq cnc1\leq c2\leq...\leq cn$, it follows that the strength of the cycle $vn-2vn-1vnvn-2vn-1vn$ is $\alpha\alpha$. Hence $u\in\{vn-2,vn-1,vn\}u\in\{vn-2,vn-1,vn\}$...(1). To prove $cn-3<cn-2cn-3<cn-2$, suppose not. That is, $cn-3=cn-2cn-3=cn-2$. Then $C1=vnvn-1vn-2C1=vnvn-1vn-2$ and $C2=vnvn-1vn-3C2=vnvn-1vn-3$ have the same strength, and hence the removal of $vn-2,vn-1,vn-2,vn-1$, or $vnvn$ will not reduce $CC(G)CC(G)$, which is a contradiction to (1). Hence, $cn-3<cn-2cn-3<cn-2$. Conversely, suppose that $cn-3<cn-2cn-3<cn-2$. To prove GG has a cyclic cut node. Since $cn\geq cn-1\geq cn-2cn\geq cn-1\geq cn-2$ and $cn-2>cn-3cn-2>cn-3$, all cycles of GG have strength less than that of the strength of $vnvn-1vn-2vnvn-1vn-2$. Hence, the deletion of $vn,vn-1,vn,vn-1$, or $vn-2vn-2$ will reduce the cycle connectivity of GG. Hence, $vn,vn-1,vn,vn-1$, and $vn-2vn-2$ are cyclic cut nodes of GG.

Theorem 3.1.7: For a complete fuzzy graph GG, $kc(G)\leq k(G)kc(G)\leq k(G)$.

Proof: Given a complete fuzzy graph GG with vertices $v1,v2,...,vnv1,v2,...,vn$ such that $ds(v1)\leq ds(v2)\leq...\leq ds(vn)ds(v1)\leq ds(v2)\leq...\leq ds(vn)$. Let $v1v1$ be a vertex such that $ds(v1)=\delta s(G)ds(v1)=\delta s(G)$.

4. Conclusion

The concept of connectivity plays a vital role in fuzzy graph theory, as it represents the relationships between the objects in a given set. Fuzzy graphs are particularly valuable in modeling real-time systems where the level of information varies with different levels of precision.

In this paper, we have analyzed the criteria for the connectivity of a fuzzy graph. We have also explored cyclic vertex connectivity and cyclic edge connectivity, strong connected domination, and node connectivity in fuzzy graphs. These discussions provide a deeper understanding of how fuzzy graphs can be used to represent complex systems with varying levels of detail and accuracy, highlighting their applicability in diverse real-world scenarios.

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